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K_{l4} decays

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Abstract. An effective theory of large-N_C QCD of pseudoscalar, vector, and axial-vector mesons has been used to study six K_{l4} decay modes. It has been found that the matrix elements of the axial-vector current dominate the K_{l4} decays. PCAC is satisfied. A relationship between three form factors of the axial-vector current has been predicted. Partial-wave analysis has been done. Non-zero phase shifts are originated in $\rho \to \pi\pi$. The decay rates are calculated in the chiral limit. In this study there is no adjustable parameter.

PACS. 13.25.-R Hadronic decays of mesons

1 Introduction

There is rich physics in kaon decays. Study on rare kaon decays is still active. The theoretical study of K_{l4} decays has a long history [1–5]. In the sixties the current algebra [2] has been applied to evaluate the form factors of K_{l4}. In the nineties the form factors have been calculated to next-to-leading order in the Chiral Perturbation Theory(ChPT) [3]. By combining the phenomenologies of K_{l4} and $\pi\pi$ scattering the three parameters $\mathcal{L}_{1,2,3}$ of ChPT have been determined. In ref. [4] the theoretical uncertainties affecting the Pais-Treiman method have been investigated and it was found that the corrections to the Pais-Treiman formula from neglecting higher partial waves are less than 1%. In order to obtain these results in the ChPT the form factors at the one-loop level have been used. A study of measuring the $\pi\pi$ phase shifts in K_{e4} decay has been presented in ref. [5].

As pointed in ref. [3] there is a puzzle in the studies of K_{l4}. Using as input the experimental central values of the form factors [6], the total decay rate of $K^+ \to \pi^+\pi^- e^+\nu_e$ is determined to be [3]

$$\Gamma_{K_{e4}} = 2.94 \times 10^3\,\mathrm{s}^{-1}$$

and the experimental value [6] is

$$\Gamma_{K_{e4}} = (3.16 \pm 0.14) \times 10^3\,\mathrm{s}^{-1}.$$

In this paper we use a different approach to study K_{l4} decays. In ref. [7] we have proposed an effective theory of large-N_C QCD [8] of pseudoscalar, vector, and axial-vector mesons. In this theory the diagrams at the tree level are at the leading order in the large-N_C expansion and the loop diagrams of mesons are at higher orders. So far, all calculations are done at the tree level and the results show that this theory is phenomenologically successful [9–11].

We have used this theory to study K_{l3} [7], $K \to e\nu\gamma$ [9, 12], kaon form factors [11], , πK scattering [10], and $\pi\pi$ scattering [7]. Theoretical results agree well with the data. This theory extends the study of meson physics to higher energy. It takes the ChPT as the low-energy limit. All the ten parameters of ChPT have been predicted [12]. Theoretical values of these parameters are compatible with the ones determined by the input data in the ChPT [13]. In this theory the Vector Meson Dominance(VMD) is a natural result and PCAC is satisfied. There are five parameters: three current quark masses, a parameter related to the quark condensate, and a universal coupling constant g which is determined to be 0.39 by fitting $\rho \to ee^+$. All parameters have been fixed by previous studies.

In this paper we use this theory of pseudoscalar, vector, and axial vector mesons [7] to study $K^- \to \pi^+\pi^- l\nu, \pi^0\pi^0 l\nu$, and $K_L \to \pi^\pm\pi^0 l^\mp\nu$. There is no adjustable parameter.

The Lagrangian of this theory [7] is

$$\begin{aligned}\mathcal{L} = &\bar{\psi}(x)(i\gamma\cdot\partial + \gamma\cdot v + \gamma\cdot a\gamma_5 - mu(x))\psi(x)\\ &+\frac{1}{2}m_1^2(\rho_i^\mu\rho_{\mu i} + \omega^\mu\omega_\mu + a_i^\mu a_{\mu i} + f^\mu f_\mu)\\ &+\frac{1}{2}m_2^2(K_\mu^{*a}\bar{K}^{*a\mu} + K_1^\mu K_{1\mu}) + \frac{1}{2}m_3^2(\phi_\mu\phi^\mu + f_s^\mu f_{s\mu})\\ &+\bar{\psi}(x)_L\gamma\cdot W\psi(x)_L + \mathcal{L}_W + \mathcal{L}_{\mathrm{lepton}} - \bar{\psi}M\psi, \qquad (1)\end{aligned}$$

where $a_\mu = \tau_i a_\mu^i + \lambda_a K_{1\mu}^a + (\frac{2}{3} + \frac{1}{\sqrt{3}}\lambda_8)f_\mu + (\frac{1}{3} - \frac{1}{\sqrt{3}}\lambda_8)f_{s\mu}$ ($i = 1,2,3$ and $a = 4,5,6,7$), $v_\mu = \tau_i\rho_\mu^i + \lambda_a K_\mu^* + (\frac{2}{3} + \frac{1}{\sqrt{3}}\lambda_8)\omega_\mu + (\frac{1}{3} - \frac{1}{\sqrt{3}}\lambda_8)\phi_\mu$, W_μ^i is the W-boson, and $u = \exp\{\gamma_5 i(\tau_i\pi_i + \lambda_a K^a + \eta + \eta')\}$, m is a parameter, and M is the mass matrix of u, d, s quarks, the masses m_1^2, m_2^2, and m_3^2 have been determined theoretically. The

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introduction of physical meson fields, of the universal coupling constant g, and of other physical quantities can be found in ref. [7]. We start from this Lagrangian to study K_{l4} decays.

The amplitudes of the vector and axial-vector currents of K_{l4} decays are expressed as

$$\begin{aligned}\langle\pi^i\pi^j|A_\mu|K\rangle &= \frac{1}{2\sqrt{8m_K\omega_1\omega_2}}\frac{i}{m_K}\\ &\times\{(p_1+p_2)_\mu F^{ij}+(p_1-p_2)_\mu G^{ij}+q_\mu R^{ij}\},\\ \langle\pi^i\pi^j|V_\mu|K\rangle &= \frac{1}{2\sqrt{8m_K\omega_1\omega_2}}\frac{H^{ij}}{m_K^3}\\ &\times\varepsilon^{\mu\nu\lambda\rho}p_\nu(p_1+p_2)_\lambda(p_1-p_2)_\rho, \qquad (2)\end{aligned}$$

where p_1, p_2, p are the momenta of two pions and of the kaon, respectively, $q = p - p_1 - p_2$, and $i, j = +, -, 0$. We define

$$q_1^2 = (p-p_1)^2, \quad q_2^2 = (p-p_2)^2, \quad q_3^2 = (p_1+p_2)^2.$$

The form factors, F^{ij}, G^{ij}, R^{ij} and H^{ij} are functions of q^2, q_1^2, q_2^2, and q_3^2. These four variables satisfy

$$q_1^2+q_2^2+q_3^2 = m_K^2+2m_\pi^2+q^2.$$

The paper is organized as: 1) introduction; 2) isospin relation; 3) form factors of the vector current; 4) $K^*\to K\pi\pi$ decay; 5) form factors of axial-vector current; 6) decay rates; 7) conclusions.

2 Isospin relation

For the decay modes $K^-\to\pi^+\pi^- l\nu, \pi^0\pi^0 l\nu$ and $\bar{K^0}\to\pi^+\pi^0 l\nu$ there are isospin relations between the form factors. We take $-\pi^+$, π^0, and π^- as isospin triplet and $-\bar{K^0}$ and K^- as isospin doublet. The isospin relation is obtained as

$$A^{+-} = A^{00} - \frac{1}{\sqrt{2}}A^{+0}, \qquad (3)$$

where $A^{ij} = F^{ij}, G^{ij}, R^{ij}, H^{ij}$, respectively.

3 Form factors of the vector current

The VMD is revealed from this theory [7]. The coupling between the W-bosons and the bosonized vector current($\Delta s = 1$) has been derived as [9]

$$\begin{aligned}\mathcal{L}^V = \frac{g_W}{4}\sin\theta_C g\{&-\frac{1}{2}(\partial_\mu W_\nu^+-\partial_\nu W_\mu^+)(\partial_\mu K_\nu^{*-}-\partial_\nu K_\mu^{*-})\\ &-\frac{1}{2}(\partial_\mu W_\nu^- -\partial_\nu W_\mu^-)(\partial_\mu K_\nu^{*+}-\partial_\nu K_\mu^{*+})\\ &+W_\mu^+ j_\mu^- + W_\mu^- j_\mu^+\}, \qquad (4)\end{aligned}$$

where $j_\mu^\pm$ is obtained by substituting

$$K_\mu^\pm \to \frac{g_W}{4}\sin\theta_C g W_\mu^\pm$$

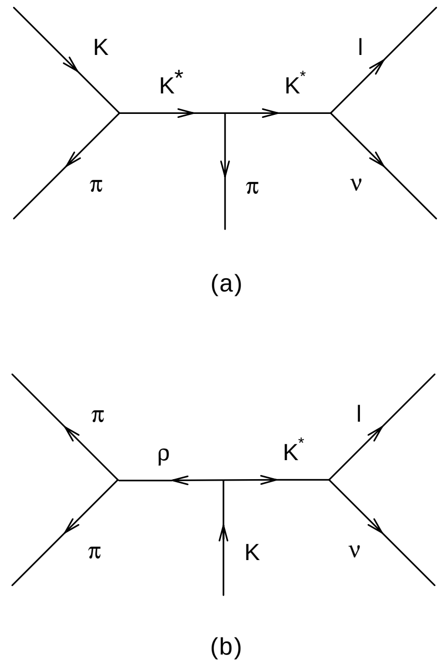

Fig. 1. Feynman diagrams of the vector current.

into the vertex in which the field of the K^*-meson, K_μ, is involved.

The matrix elements of the vector current of K_{l4} can be calculated by using eq. (4). There are two subprocesses which are shown in fig. 1(a,b). Three kinds of vertices are involved: the contact term $\mathcal{L}_{K^*K\pi\pi}$, $\mathcal{L}_{K^*K^*\pi}$ and $\mathcal{L}_{K^*K\pi}$, and $\mathcal{L}_{K^*K\rho}$ and $\mathcal{L}_{\rho\pi\pi}$. In the chiral limit, $m_q\to 0$, all these vertices have been derived from the Lagrangian (1) [7] and are listed below:

$$\begin{aligned}\mathcal{L}_{K^*K^*\pi} &= -\frac{N_C}{\pi^2 g^2 f_\pi}\,\varepsilon^{\mu\nu\alpha\beta}d_{aci}K_\mu^a\partial_\nu K_\alpha^c\partial_\beta\pi^i,\\ \mathcal{L}_{K^*K\pi} &= \frac{2}{g}f(q^2)f_{abi}K_\mu^a(\partial_\mu\pi^i K^b-\pi^i\partial_\mu K^b),\\ f(q^2) &= 1+\frac{q^2}{2\pi^2 f_\pi^2}[(1-\frac{2c}{g})^2-4\pi^2c^2)],\\ c &= \frac{f_\pi^2}{2gm_\rho^2},\\ \mathcal{L}_{K^*\rho K} &= -\frac{N_C}{\pi^2 g^2 f_\pi^2}\varepsilon^{\mu\nu\alpha\beta}d_{abi}K_\mu^a\partial_\nu\rho_\alpha^i\partial_\beta K^b,\\ \mathcal{L}_{\rho\pi\pi} &= \frac{2}{g}f(q^2)\epsilon_{ijk}\rho_\mu^i\pi^j\partial_\mu\pi^k,\\ \mathcal{L}_{K^*K\pi\pi} &= \frac{2}{g\pi^2 f_\pi^3}(1-\frac{6c}{g}+\frac{6c^2}{g^2})\\ &\times d_{abe}f_{cde}\varepsilon^{\mu\nu\alpha\beta}K_\mu^a\partial_\nu P^b\partial_\alpha P^c\partial_\beta P^d. \qquad (5)\end{aligned}$$

The matrix element of the vector current of $K_{l4}(2)$ are calculated by using eqs. (4),(5). The form factors H^{ij} are found:

$$H^{+-}=\frac{2m_K^3m_{K^*}^2}{q^2-m_{K^*}^2}\Big\{\frac{1}{\pi^2f_\pi^3}(1-\frac{6c}{g}+\frac{6c^2}{g^2})-\frac{N_C}{g^2\pi^2f_\pi}\frac{f(q_2^2)}{q_2^2-m_{K^*}^2}-\frac{N_C}{2g^2\pi^2f_\pi}\frac{f(q_3^2)}{q_3^2-m_\rho^2+i\sqrt{q_3^2}\Gamma(q_3^2)}\Big\}, \quad (6)$$

$$H^{00}=-\frac{2m_K^3m_{K^*}^2}{q^2-m_{K^*}^2}\frac{N_c}{2g^2\pi^2f_\pi}\Big\{\frac{f(q_2^2)}{q_2^2-m_{K^*}^2}-\frac{f^2(q_1^2)}{q_1^2-m_{K^*}^2}\Big\}, \quad (7)$$

$$H^{+0}=\frac{\sqrt{2}m_K^3m_{K^*}^2}{q^2-m_{K^*}^2}\Big\{-\frac{2}{\pi^2f_\pi^3}(1-\frac{6c}{g}+\frac{6c^2}{g^2})+\frac{N_C}{\pi^2g^2f_\pi}[\frac{f(q_1)}{q_1^2-m_{K^*}^2}+\frac{f(q_2)}{q_2^2-m_{K^*}^2}+\frac{f(q_3)}{q_3^2-m_\rho^2+i\sqrt{q_3^2}\Gamma_\rho(q_3^2)}]\Big\}, \quad (8)$$

where Γ_ρ is the decay width of the ρ-meson

$$\Gamma_\rho(q_3^2)=\frac{\sqrt{q_3^2}f^2(q_3^2)}{12g^2\pi}(1-\frac{4m_\pi^2}{q_3^2})^{\frac{3}{2}}. \quad (9)$$

The decay width of the ρ-meson is determined to be 142 MeV which is in good agreement with the data. Equations (6)-(8) show that the isospin relation(3) is satisfied.

The form factors (6)-(8) originate in the Wess-Zumino-Witten anomaly. They are different from the ones presented in ref. [3]. These form factors are responsible for the decay $K^*\to K\pi\pi$.

4 K* → Kππ decay

The form factors of the vector current are determined by the vertices (5). On the other hand, these vertices are responsible for the decay of $K^*\to K\pi\pi$. As a test of these vertices the decay widths of $K^*\to K\pi\pi$ are calculated:

$$\Gamma(K^{*-}\to K^-\pi^+\pi^-)=\frac{1}{96(2\pi)^3m_{K^*}}\int dk_1^2dk_2^2\{p_1^2p_2^2-(\mathbf{p}_1\cdot\mathbf{p}_2)^2\}|A|^2=0.29\times10^{-5}\,\mathrm{GeV}\,, \quad (10)$$

which is less than the experimental upper limit [14], where A is the decay amplitude and is determined by eq. (5):

$$A=\frac{4}{g\pi^2f_\pi^3}(1-\frac{6c}{g}+\frac{6c^2}{g^2})-\frac{4N_c}{g^3\pi^2f_\pi}\frac{f(k_2^2)}{k_2^2-m_{K^*}^2+i\sqrt{k_2^2}\Gamma_{K^*}(k_2^2)}-\frac{2N_c}{g^3\pi^2f_\pi}\frac{f(k_3^2)}{k_3^2-m_\rho^2+i\sqrt{k_3^2}\Gamma_\rho(k_3^2)}\,, \quad (11)$$

where $k_1^2=(p+p_1)^2$, $k_2^2=(p+p_2)^2$, $k_3^2=(p_1+p_2)^2$, and p_1,p_2,p are the momenta of π^+,π^- and K^-, respectively, Γ_{K^*} is the decay width of K^*:

$$\Gamma_{K^*}(k_2^2)=\frac{f^2(k_2^2)}{2\pi g^2k_2^2}\{\frac{1}{4k_2^2}(k_2^2+m_K^2-m_\pi^2)^2-m_K^2\}^{\frac{3}{2}}. \quad (12)$$

Using eq. (5), we obtain

$$\Gamma(K^{*-}\to K^-\pi^0\pi^0)=\frac{1}{192(2\pi)^3m_{K^*}}\int dk_1^2dk_2^2\{p_1^2p_2^2-(\mathbf{p}_1\cdot\mathbf{p}_2)^2\}\times\frac{36}{\pi^4g^6f_\pi^2}\Big\{\frac{f(k_1)}{k_1^2-m_{K^*}^2+i\sqrt{k_1^2}\Gamma_{K^*}(k_1^2)}-\frac{f(k_2)}{k_2^2-m_{K^*}^2+i\sqrt{k_2^2}\Gamma_{K^*}(k_2^2)}\Big\}^2=0.61\times10^{-6}\,\mathrm{GeV} \quad (13)$$

and

$$\Gamma(K^{*-}\to \bar{K}^0\pi^-\pi^0)=\frac{1}{96(2\pi)^3m_{K^*}}\int dk_1^2dk_2^2\{p_1^2p_2^2-(\mathbf{p}_1\cdot\mathbf{p}_2)^2\}|B|^2=0.38\times10^{-4}\,\mathrm{GeV}, \quad (14)$$

where

$$B=-\frac{8}{\sqrt{2}gf_\pi^3}(1-\frac{6c}{g}+\frac{6c^2}{g^2})+\frac{12}{\sqrt{2}\pi^2g^3f_\pi}\Big\{\frac{f(k_1)}{k_1^2-m_{K^*}^2+i\sqrt{k_1^2}\Gamma_{K^*}(k_1^2)}+\frac{f(k_2)}{k_2^2-m_{K^*}^2+i\sqrt{k_2^2}\Gamma_{K^*}(k_2^2)}+\frac{f(k_3)}{k_3^2-m_\rho^2+i\sqrt{k_3^2}\Gamma_\rho(k_3^2)}\Big\}. \quad (15)$$

The theoretical results are compatible with the data [14].

5 Form factors of the axial-vector current

In the chiral limit, the axial-vector part of the interaction between the W-boson and mesons is expressed as [9]

$$\mathcal{L}^{As}=\frac{g_W}{4}\frac{1}{f_a}\sin\theta_C\times\Big\{-\frac{1}{2}(\partial_\mu W_\nu^\pm-\partial_\nu W_\mu^\pm)(\partial^\mu K_1^{\mp\nu}-\partial^\nu K_1^{\mp\mu})+W^{\pm\mu}j_\mu^\mp\Big\}+\frac{g_W}{4}\sin\theta_C\Delta m^2f_aW_\mu^\pm K_1^{\mp\mu}+\frac{g_w}{4}\sin\theta_C f_KW_\mu^\pm\partial^\mu K^\mp, \quad (16)$$

where K_1 is the axial-vector kaon field and $j_\mu^\pm$ are obtained by substituting $K_{1\mu}^\pm\to\frac{g_W}{4f_a}\sin\theta_C W_\mu^\pm$ into the vertex in which K_1 fields are involved. From ref. [9] we have

$$f_a=g^{-1}(1-\frac{1}{2\pi^2g^2})^{-\frac{1}{2}}, \quad (17)$$

$$\Delta m^2=6m^2g^2=f_\pi^2(1-\frac{f_\pi^2}{g^2m_\rho^2})^{-1}. \quad (18)$$

The mass of the K_1-meson is determined [7]:

$$(1-\frac{1}{2\pi^2g^2})m_{K_1}^2=6m^2+m_{K^*}^2. \quad (19)$$

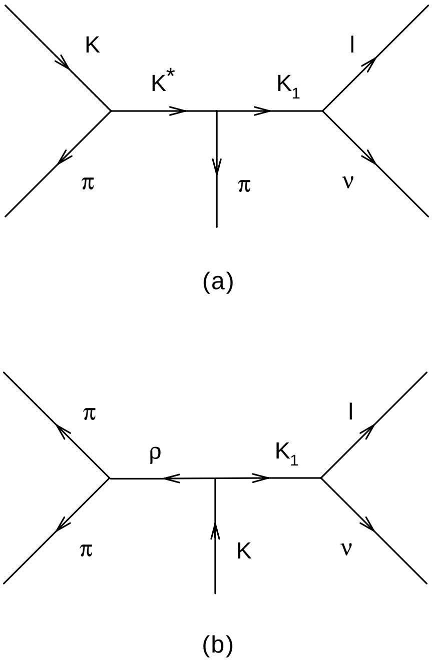

Fig. 2. Feynman diagrams of the axial-vector current.

The numerical value is $m_{K_1} = 1.322\,\text{GeV}$ which is compatible with the data [14].

Two subprocesses contribute to the matrix element of the axial-vector current. They are shown in fig. 2(a,b). The vertices of mesons involved in these processes are $\mathcal{L}_{K_1K^*\pi}$, $\mathcal{L}_{K^*K\pi}$ and $\mathcal{L}_{K_1\rho K}$, $\mathcal{L}_{\rho\pi\pi}$. There is a contact term $\mathcal{L}_{K_1K\pi\pi}$ too. However, the calculation shows that the contribution of the contact term is very small and negligible. In the chiral limit, these vertices have been derived from the Lagrangian (1) [7]

$$\begin{aligned}\mathcal{L}_{K_1K^*\pi} &= f_{abi}\{A(p^2)K^a_{1\mu}K^{*b}_\mu\pi^i \\ &\quad -BK^a_{1\mu}K^{*b}_\nu\partial_{\mu\nu}\pi^i + DK^a_{1\mu}\partial^\mu(K^{*b}_\nu\partial^\nu\pi^i)\}, \quad (20)\\ \mathcal{L}_{K_1\rho K} &= -f_{abi}\{A(p^2)K^a_{1\mu}\rho^i_\mu K^b \\ &\quad -BK^a_{1\mu}\rho^i_\nu\partial_{\mu\nu}K^b + DK^a_{1\mu}\partial^\mu(\rho^i_\nu\partial^\nu K^b)\}, \quad (21)\end{aligned}$$

where

$$\begin{aligned}A(p^2) &= \frac{2}{f_\pi}gf_a\{\frac{F^2}{g^2} + p^2[\frac{2c}{g} + \frac{3}{4\pi^2g^2}(1-\frac{2c}{g})] \\ &\quad + q^2[\frac{1}{2\pi^2g^2} - \frac{2c}{g} - \frac{3}{4\pi^2g^2}(1-\frac{2c}{g})]\}, \quad (22)\\ F^2 &= f_\pi^2(1-\frac{2c}{g})^{-1}, \quad (23)\\ B &= -\frac{2}{f_\pi}gf_a\frac{1}{2\pi^2g^2}(1-\frac{2c}{g}), \quad (24)\\ D &= -\frac{2}{f_\pi}f_a\{2c + \frac{3}{2\pi^2g}(1-\frac{2c}{g})\}, \quad (25)\end{aligned}$$

where q and p are the momenta of K_1 and of the vector meson, respectively.

By using eqs. (16,20,21), we obtain

$$\begin{aligned}\langle\pi^+\pi^-|A_\mu|K^-\rangle &= \frac{1}{\sqrt{2}}(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu})\frac{g^2f_am^2_{K^*}}{q^2-m^2_{K_1}} \\ &\quad \times\langle\pi^+\pi^-|\{A(p_{K^*})\bar{K^0}_\nu\pi^- - B\bar{K^0}_\lambda\partial_{\lambda\nu}\pi^-\} \\ &\quad -\frac{1}{\sqrt{2}}\{A(p_\rho)\rho^0_\nu K^- - B\rho^0_\lambda\partial_{\lambda\nu}K^-\}|K^-\rangle. \quad (26)\end{aligned}$$

Equation (26) shows that PCAC is satisfied in the chiral limit. The reason is that the Lagrangian (1) is chiral symmetric in the limit $m_q \to 0$. On the other hand, the PCAC results in the cancellations between the four terms of eq. (26). Equation (16) shows that the axial-vector current has a more complicated structure than the vector current (4) does. In the chiral limit applying PCAC to eq. (2), a relationship between the three form factors of the axial-vector current is obtained:

$$R = -\frac{1}{q^2}\{q\cdot(p_1+p_2)F + q\cdot(p_1-p_2)G\}. \quad (27)$$

Substituting the vertices (20,21) into eq. (26), the three form factors are obtained:

$$\begin{aligned}F^{+-} &= \frac{2gf_am^2_{K^*}m_K}{q^2-m^2_{K_1}} \\ &\quad \times\{\frac{f(q_2^2)}{q_2^2-m^2_{K^*}}[\frac{3}{2}A(q_2^2) + \frac{1}{2}Bp_1\cdot(p+p_2)] \\ &\quad + \frac{f(q_3^2)}{q_3^2-m_\rho^2+i\sqrt{q_3^2}\Gamma_\rho(q_3^2)}Bp\cdot(p_2-p_1)\}, \quad (28)\\ G^{+-} &= -\frac{2gf_am^2_{K^*}m_K}{q^2-m^2_{K_1}} \\ &\quad \times\{\frac{f(q_2^2)}{q_2^2-m^2_{K^*}}[-\frac{1}{2}A(q_2^2) + \frac{1}{2}Bp_1\cdot(p+p_2)] \\ &\quad - \frac{f(q_3^2)}{q_3^2-m_\rho^2+i\sqrt{q_3^2}\Gamma_\rho(q_3^2)}A(q_3^2)\}. \quad (29)\end{aligned}$$

In the same way the form factors of other two decay modes are obtained

$$\begin{aligned}F^{00} &= \frac{gf_am^2_{K^*}m_K}{q^2-m^2_{K_1}} \\ &\quad \times\{\frac{f(q_1^2)}{q_1^2-m^2_{K^*}}[\frac{3}{2}A(q_1^2) + \frac{1}{2}B(p_2\cdot p + p_2\cdot p_1)] \\ &\quad + \frac{f(q_2^2)}{q_2^2-m^2_{K^*}}[\frac{3}{2}A(q_2^2) + \frac{1}{2}B(p_1\cdot p + p_1\cdot p_2)]\}, \quad (30)\\ G^{00} &= -\frac{gf_am^2_{K^*}m_K}{q^2-m^2_{K_1}} \\ &\quad \times\{\frac{f(q_1^2)}{q_1^2-m^2_{K^*}}[\frac{1}{2}A(q_1^2) - \frac{1}{2}B(p_2\cdot p + p_2\cdot p_1)] \\ &\quad + \frac{f(q_2^2)}{q_2^2-m^2_{K^*}}[-\frac{1}{2}A(q_2^2) + \frac{1}{2}B(p_1\cdot p + p_1\cdot p_2)]\}, \quad (31)\end{aligned}$$

$$F^{+0} = \frac{\sqrt{2}gf_a m^2_{K^*}m_K}{q^2 - m^2_{K_1}} \times\Big\{\frac{f(q_1^2)}{q_1^2 - m^2_{K^*}}[\frac{3}{2}A(q_1^2) + \frac{1}{2}B(p_2\cdot p + p_2\cdot p_1)] - \frac{f(q_2^2)}{q_2^2 - m^2_{K^*}}[\frac{3}{2}A(q_2^2) + \frac{1}{2}B(p_1\cdot p + p_1\cdot p_2)] + \frac{2f(q_3^2)}{q_3^2 - m_\rho^2 + i\sqrt{q_3^2}\Gamma_\rho(q_3^2)}Bp\cdot(p_1 - p_2)\Big\}, \quad (32)$$

$$G^{+0} = -\frac{\sqrt{2}gf_a M^2_{K^*}m_K}{q^2 - m^2_{K_1}} \times\Big\{\frac{f(q_1^2)}{q_1^2 - m^2_{K^*}}[\frac{1}{2}A(q_1^2) - \frac{1}{2}B(p_2\cdot p + p_2\cdot p_1)] - \frac{f(q_2^2)}{q_2^2 - m^2_{K^*}}[-\frac{1}{2}A(q_2^2) + \frac{1}{2}B(p_1\cdot p + p_1\cdot p_2)] + \frac{2f(q_3^2)}{q_3^2 - m_\rho^2 + i\sqrt{q_3^2}\Gamma_\rho(q_3^2)}A(q_3^2)\Big\}. \quad (33)$$

The isospin relations (3) among these form factors are satisfied.

The partial-wave analysis of these form factors can be done. The decay channel $\rho\to\pi\pi$ contributes to the decay modes of $\pi^+\pi^-$ and $\pi^+\pi^0$. The range of the variable q_3^2 is $4m_\pi^2 < q_3^2 < (m_K - m_l)^2$ in which the decay width $\Gamma_\rho(q_3^2)$ is not zero. The form factors, A^{+-} and A^{+0} are complex functions of q_3^2. The $\rho\to\pi\pi$ does not contribute to $\pi^0\pi^0$ mode. Therefore, F^{00} and G^{00} are real. The q_1^2 and q_2^2 variables are expressed as

$$q_1^2 = \frac{1}{2}(m_K^2+2m_\pi^2+q^2-q_3^2) + (1-\frac{4m_\pi^2}{q_3^2})^{\frac{1}{2}}X\cos\theta_\pi, \quad (34)$$

$$q_2^2 = \frac{1}{2}(m_K^2+2m_\pi^2+q^2-q_3^2) - (1-\frac{4m_\pi^2}{q_3^2})^{\frac{1}{2}}X\cos\theta_\pi, \quad (35)$$

where $X = \{\frac{1}{4}(m_K^2-q^2-q_3^2)^2-q^2q_3^2\}^{\frac{1}{2}}$ and θ_π is the angle between $\mathbf{p}_1$ and $\mathbf{p}$ in the rest frame of the two pions.

In refs. [6,15] by assuming the s- and p-waves dominance the partial-wave analysis has been done. As pointed in ref. [3], there are partial waves of higher orders. This is true in the form factors obtained in this paper. In order to compare with data we expand the form factors (28-33) up to s- and p-wave only. We obtain

$$F^{+-} = F_s^{+-} + F_p^{+-}e^{i\delta_p^{+-}}\cos\theta_\pi, \quad G^{+-} = G_s^{+-}e^{i\delta_s^{+-}} + G_p^{+-}\cos\theta_\pi, \quad (36)$$

$$F^{+0} = F_p^{+0}e^{i\delta_p^{+0}}\cos\theta_\pi, \quad G^{+0} = G_s^{+0}e^{\delta_s^{+0}}, \quad (37)$$

$$F^{00} = F_s^{00}, \quad G^{00} = G_p^{00}\cos\theta_\pi. \quad (38)$$

All the phase shifts are caused by the decay $\rho\to\pi\pi$ and they are functions of q^2 and q_3^2.

In order to compare with the data $s_l = q^2$ and $s_\pi = q_3^2$ are used in following equations. For the $\pi^+\pi^-$ mode the numerical results are

$$F_s^{+-} = 4.36\{1 + 0.12\frac{s_l}{4m_\pi^2} - 0.16(\frac{s_\pi}{4m_\pi^2} - 1)\},$$
$$G_s^{+-} = 5.06\{1 + 0.002\frac{s_l}{4m_\pi^2} + 0.23(\frac{s_\pi}{4m_\pi^2} - 1)\},$$
$$H_s^{+-} = -5.82\{1 + 0.13\frac{s_l}{4m_\pi^2} + 0.023(\frac{s_\pi}{4m_\pi^2} - 1)\}, \quad (39)$$

For the $\pi^+\pi^0$ mode we obtain

$$F_s^{+0} = 0,\quad F_p^{+0} = 1.1,\quad G^{+0} = 9.04\{1 + 0.052\frac{s_l}{4m_\pi^2} + 0.028(\frac{s_\pi}{4m_\pi^2} - 1)\}\quad H^{+0} = 5.78. \quad (40)$$

The corresponding amplitudes for the $\pi^0\pi^0$ mode can be obtained by the isospin relations (3).

In refs. [6,15] assumptions, like the absence of higher waves, s_l-independence of the form factors and equality of the slopes, have been made. Under these assumptions the following expressions for the $\pi^+\pi^-$ mode have been determined in ref. [6]:

$$F_s^{+-} = (5.59\pm0.14)\{1 + \lambda_f(\frac{s_\pi}{4m_\pi^2} - 1)\},$$
$$G^{+-} = (4.77\pm0.27)\{1 + \lambda_g(\frac{s_\pi}{4m_\pi^2} - 1)\},$$
$$H^{+-} = -(2.68\pm0.68)\{1 + \lambda_h(\frac{s_\pi}{4m_\pi^2} - 1)\},$$
$$\lambda_f = \lambda_g = \lambda_h = 0.08\pm0.02. \quad (41)$$

For the $\pi^+\pi^0$ mode the form factors determined in ref. [15] are

$$F = f_s e^{i\delta_s} + f_p\cos\theta_\pi e^{i\delta_p},\quad G = ge^{i\delta_p},\quad H = he^{i\delta_p},\quad g = (7.8\pm0.7\pm0.2)\times\{1 + (0.014\pm0.087\pm0.070)(\frac{s_\pi}{4m_\pi^2} - 1)\}, \quad (42)$$

$$\frac{f_s}{g} = -0.010\pm0.016\pm0.017,\quad \frac{f_p}{g} = -0.079\pm0.049\pm0.022,\quad \frac{h}{g} = -0.017\pm0.31\pm0.31. \quad (43)$$

A comparison between the theoretical results of the form factors and the measurements shows that

1) for $\pi^+\pi^-$ mode the central values of the theoretical form factors (multiplied by 2), F and G, are compatible with the data (41),
2) the dependencies of F_s^{+-}(39) on s_l and s_π are different from the data (41),
3) the dependence of G^{+-}(39) on s_l and s_π is compatible with the data (41),

4) the central value of H^{+-}(39) is greater than the data (41) by a factor of two,
5) from eqs. (40) we obtain

$$g = 9.04, \quad \frac{f_s}{g} = 0, \quad \frac{f_p}{g} = 0.12, \quad \frac{h}{g} = 0.64,$$

the central values of g, $\frac{f_s}{g}$, $\frac{f_p}{g}$ are compatible with the data (43),
6) the theoretical value of $\frac{h}{g}$ is different from the data (43),
7) the dependence of G_s^{+-} on s_l and s_π is different from the data (43).
8) In ref. [6] F_p^{+-} was found to be compatible with zero, and hence we put it equal to zero when the final result of G^{+-} was derived. In this paper the calculation of the form factors shows that the values of G^{+-} are greater than F^{+-} by more than a factor of two. The calculation also shows that F_p^{+-} is less than 10% of F_s^{+-}. These results are in agreement with the data [6].
9) In this theory the phase shift of the p-wave originates in the decay $\rho \to \pi\pi$, which is a function of s_l and s_π. The value of this phase-shift is about few degrees. The phase shift of the s-wave cannot be obtained in this theory. As pointed out in ref. [7], in order to get the s-wave phase-shift the $\pi\pi$ scattering the σ-meson has to be introduced into this theory.

It is necessary to point out that because of kinematic reasons, the contribution of the anomalous form factor H is negligibly small and theoretical H fits the data of the decay $K^* \to K\pi\pi$ well.

For reference the form factor obtained by Weinberg [2] are listed below

$$F = G = \frac{m_K\sqrt{2}}{f_\pi} = 7.48,$$
$$H = 0. \tag{44}$$

6 Decay rates

The decay rates of the three modes of K_{e4} and $K_{\mu4}$ are calculated. As mentioned above, all the form factors are derived in the chiral limit. Therefore, only the leading terms of the masses of the kaon and pions are kept in the calculation of the decay rates.

Ignoring m_e, only the form factors F, G, and H contribute to the decay rates of K_{e4}. By using the formula of ref. [1], we obtain

$$\Gamma(K^- \to \pi^+\pi^- e\nu) = 3130\,\mathrm{s}^{-1},$$

$$\Gamma(K^- \to \pi^0\pi^0 e\nu) = 0.221\times10^{-21}\ \mathrm{GeV}, \quad B = 0.42\times10^{-5}.$$

$$\Gamma(K_L \to \pi^+\pi^0 e\nu) = 4923\,\mathrm{s}^{-1}, \quad B = 2.55\times10^{-4}.$$

The experimental data are

$$\Gamma(K^- \to \pi^+\pi^- e\nu) = (3160 \pm 140)\,\mathrm{s}^{-1}\ [6],$$

$$B(\pi^0\pi^0) = (2.54 \pm 0.89)\times10^{-5}(10\ \text{events})\ [16],$$

$$\Gamma(K_L \to \pi^+\pi^0 e\nu) = (1700 \pm 320)\,\mathrm{s}^{-1}\ [15],$$

$$B(\pi^-\pi^0) = (6.2 \pm 2.0)\times10^{-5}\ [17],$$

$$B(\pi^-\pi^0) < 200\times10^{-5}\ [18].$$

The theoretical result of the $\pi^+\pi^-$ mode agrees well with the data. For the $\pi^+\pi^0$ mode, theory is greater than the experiment by more than a factor of two. However, the form factors of the axial-vector current are compatible with the data [15].

Using the full expressions of the form factors(28-33) and the expansions (39),(40) the puzzle raised in ref. [3] can be studied. By using the expansion (39),(40) the decay widths of $K^- \to \pi^+\pi^- e\nu$ and $K_L \to \pi^+\pi^0 e\nu$ are calculated to be

$$\Gamma(K^- \to \pi^+\pi^- e\nu) = 1519\,\mathrm{s}^{-1},$$
$$\Gamma(K_L \to \pi^+\pi^0 e\nu) = 2233\,\mathrm{s}^{-1}.$$

They are about half the values obtained by eqs. (28, 29, 32, 33). These results show that the expansions (39,40) may not be good approximations. The possible reason is that because of the contribution of $\rho \to \pi\pi$ the resonance factor

$$\frac{1}{q_3^2 - m_\rho^2 + \sqrt{q_3^2}\Gamma_\rho(q_3^2)}$$

appears in the form factors (28-33). The range of q_3^2 is

$$4m_\pi^2 < q_3^2 < m_K^2.$$

Therefore, use of linear expansion of q_3^2 is not a good approximation for the resonance factor. A different reason of the puzzle has been presented in ref. [4].

The form factors of the vector current are determined by anomalous vertices. The numerical calculation shows that the contribution of the form factor H is only 0.5% of the total decay rate of $K^- \to \pi^+\pi^- e\nu$. Therefore, the axial-vector current dominates the K_{l4} decays.

As shown in fig. 2(a,b) there are two channels in K_{l4} decays. The numerical calculation of $K^- \to \pi^+\pi^- e\nu$ shows that the contribution of $\rho \to \pi\pi$(fig. 2(b)) is twice of the process, $K^* \to K\pi$, (fig. 2(a)). Only the process (fig. 2(a)) contributes to $K^- \to \pi^0\pi^0 e\nu$. Because of the Bose statistics there is an additional factor of $\frac{1}{2}$ in the formula of the decay rate of this mode. Therefore, this theory predicts a smaller decay rate for the $\pi^0\pi^0$ decay mode. On the other hand, the numerical calculation shows that the process(fig. 2(b)) is the major contributor of the decay $\bar{K}^0 \to \pi^+\pi^0 e\nu$. The theory predicts a larger branching ratio for $\bar{K}^0 \to \pi^+\pi^0 e\nu$.

All the form factors contribute to $K_{\mu4}$ decays. Equations (26,27) show that in the chiral limit PCAC is satisfied and the form factor R is determined by other two form factors, F and G. The branching ratio of $K_{\mu 4}$ provides a test of this prediction. The numerical results are

$$\Gamma(K^-\to\pi^+\pi^-\mu\nu)=0.634\times10^{-21}\,\mathrm{GeV},\quad B=1.19\times10^{-5},$$

$$\Gamma(K^-\to\pi^0\pi^0\mu\nu)=0.673\times10^{-22}\,\mathrm{GeV},\quad B=0.126\times10^{-5},$$

$$\Gamma(\bar{K}^0\to\pi^+\pi^0\mu\nu)=1.01\times10^{-21}\,\mathrm{GeV},\quad B=0.793\times10^{-4}.$$

The experimental data [14] is

$$B(K^-\to\pi^+\pi^-\mu\nu)=(1.4\pm0.9)\times10^{-5}.$$

The theory agrees well with the data.

7 Conclusions

All the four form factors of K_{l4} have been derived from an effective theory of large-N_C QCD of mesons in the chiral limit. It has been found that the contribution of the vector current is negligible and the axial-vector current is dominant in K_{l4} decays. PCAC is revealed from the theory. In the chiral limit it has been predicted that the form factor R is determined by the form factors F and G. The prediction has been tested by $K^-\to\pi^+\pi^-\mu\nu$. The theory agrees with the data. The partial-wave analysis has been done. The central values of the form factors, F and G, of the $\pi^+\pi^-$ mode are compatible with the data. The decay rate of this mode agrees well with the data too. For the $\pi^+\pi^0$ mode the form factors of the axial-vector current are compatible with the data. However, the decay rate is greater than the data. The theoretical branching ratio of $K^-\to\pi^0\pi^0 e\nu$ is less than the data. There are only 10 events. The values of the anomalous form factors originated in Wess-Zumino-Witten anomaly are greater than the data. These anomalous form factors are tested by the decays $K^*\to K\pi\pi$. The theory is consistent with the data.

In this theory $\rho\to\pi\pi$(fig. 2(b)) is the most important channel. It has been found that the linear expansion may not be a good approximation. The non-zero phase shift of p-wave originates in the decay $\rho\to\pi\pi$.

There are three problems that should be investigated in the future study of K_{l4}.

1) The s-wave phase-shift cannot be revealed from this study. The σ-meson has to be introduced in this effective theory.
2) The form factors are calculated in the chiral limit in this paper. The effects of the strange-quark in K_{l4} decays need to be studied. In previous studies [7,9–11] the amplitudes of the physical processes are calculated in the limit, $m_q=0$. We can use the comparison between theoretical results and the data to estimate the contributions of the strange-quark mass to these processes. We take the decays, $\phi\to K\bar{K}$ and $K^*\to K\pi$ as examples. Using the vertex $\mathcal{L}^{\phi K\bar{K}}$ of ref. [7], we obtain

$$\Gamma(\phi\to K^+K^-)=2.14\,\mathrm{MeV}\,,$$

$$\Gamma(\phi\to K^0\bar{K}^0)=1.4\,\mathrm{MeV}\,.$$

The data are

$$\Gamma(\phi\to K^+K^-)=2.19(1\pm0.02)\,\mathrm{MeV}\,,$$

$$\Gamma(\phi\to K^0\bar{K}^0)=1.51(1\pm0.02)\,\mathrm{MeV}.$$

The deviation is less than 10%. Using eq. (12), we obtain

$$\Gamma(K^*\to K\pi)=45.6\,\mathrm{MeV}.$$

and the data is $50.7(1\pm0.01)\,\mathrm{MeV}$. The deviation is 11%. In these calculations $g=0.39$ is used. A theoretical study on the effect of the strange-quark mass is needed. As a matter of fact, in ref. [12] the contributions of current quark masses to pseudoscalar mesons are calculated to $O(m_q^2)$ and the decay constant of pion, kaon, and η are calculated to $O(m_q)$.
3) The calculations are done at the tree level in this paper. According to ref. [7], the amplitudes of K_{l4} at the tree level are at the order of $O(N_C)$. The loop diagrams of the mesons contribute to K_{l4} decays too. The amplitudes of the loop diagrams are at $O(1)$ in the N_C expansion [7]. The loop diagrams can be calculated and the effects of loop diagrams bring modifications to the parameters determined at the tree level.

All these three problems will be investigated in the near future.

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